Head-on collision and merging entropy of black holes: reconsideration of Hawking's inequality

Masaru Siino[1]

Department of Physics, Tokyo Institute of Technology,
Oh-Okayama 1-12-1, Meguro-ku, Tokyo 152-8550, Japan

Abstract

We evaluate how much energy can be converted into gravitational radiation in head-on collision of black holes. We estimate it by the area theorem of black hole horizon incorporating merging entropy of colliding black holes from a viewpoint of black hole thermodynamics. Then we obtain an upper bound of energy rate of the gravitational radiation which is smaller than the upper bound originally derived by Hawking. The estimation correctly predicts the results of both numerical investigations in low- and high-energy head-on collision.

1 Introduction

In head-on collision of black holes, evaluation of energy converted into gravitational radiation is an important aim in astrophysical application of general relativity. The leading work is made by Hawking[1] using an area theorem of black hole event horizon, which placed an upper limit of 29% on the total energy radiated when two black holes initially at rest coalesce, in head-on collision of spin-less black holes with identical masses. Using similar argument based on the area theorem, Penrose[2] derived an upper bound of 29% for ultra-relativistic head-on collision. However numerical simulation later showed that the gravitational radiation is far less than expected. They reported true value of energy rate of the gravitational radiation around 0.1%. Indeed, the fact that the gravitational wave barely radiates, is the evidence of Hawking's upper bound, but if the upper bound were a result of sufficiently detailed evaluation, it would be expected to be closer to the numerical result and to provide any prediction. In this sense, the gravitational radiation is far from enough to feel such a mathematical analysis is useful. Why is there not much gravitational radiation in this simple head-on collision?

The essence of Hawking's discussion is the area theorem of black hole event horizon[4]. The area theorem is proved, based on the mathematical concepts about event horizon generators, which are the Raychaudhri equation and their future completeness. Since these mathematical concepts are fundamental, Hawking's bound may not be severe but will be universal independently of detailed physical situations. Indeed, for the numerical experiments[3], Hawking's bound suffers no problem but says nothing instructive. Then if we want to get any useful information from such a mathematical discussion, we should change the bound more severe by making the discussion more precise. In the present work, we will make the upper bound smaller by revising the application of the area theorem to the black hole collision.

The work of Bekenstein[7] and Hawking[4] has shown, and that of many others confirmed, that the mechanical laws governing classical systems containing black holes can be placed in analogy with those of thermodynamics. The area theorem is one of the important theorems of the black hole mechanics. Its importance have increased since it is included in the black hole thermodynamics as increasing low of the Bekenstein-Hawking entropy by the assumption that the stationary black hole possesses entropy which is a fourth of its horizon area. If even in dynamical situation the black hole entropy should be generally given by a fourth of the horizon area, thermodynamical estimation of the entropy increasing will provide a dynamical changing of the horizon area, which will be evaluated for calculating the energy rate of gravitational radiation. Then in the present study of black hole collision, we expect any contribution of additional entropy like the entropy increase of mixing for the merging process, which was not incorporated in the Hawking's upper bound[1]. The entropy increase ΔS should be added to a black hole area inequality as extra area increase ΔA of black hole by $\Delta A = 4\Delta S$ and would gives smaller upper bound of the energy

[1]E-mail:msiino@th.phys.titech.ac.jp

rate. The fact that the black hole coalescence is themodynamically invertible process like the mixing process would be consistent with the mathematical fact that black holes can coalesce but not split.

Recently in high-energy head-on collision of spin-less black holes with identical masses, in which the colliding black holes initially possess large relative velocity, a remarkable numerical investigation[5] has shown that the large amount of energy is converted into gravitational radiation. Its result suggests that in the maximally high-energy limit, the energy rate tends to 14% of total energy in extrapolation by the dependence of γ fitting the prediction[5] of linear perturbation based on the zero frequency limit and point particle approximation, that is a half of Hawking's and Penrose's bounds, and quite close to the estimation 16% of D'Eath, Payne[6] using perturbative technique. Now, our problem has become more complicated than before. Can we explain both why the energy rate of the gravitational radiation is so small in the low-energy collision and is so large in the high-energy collision?

The purpose of the present article is to explain how much gravitational radiation arises in head-on collision of black holes with low or high kinetic energy using the area theorem of black hole horizon. In the second section, we recall the original work of Hawking and give a key idea of the present study. Our discussion put a base on the issue of black hole thermodynamics. In the third section, thermodynamical investigation for merging entropy gives an evaluation of the energy rate of the gravitational radiation in the low- and high-energy head-on collision of spin-less black holes with identical masses. The final section is devoted to conclusion and discussions.

2 Hawking's bound

When two black holes coalesce, the amount of energy converted into the gravitational radiation is restricted by Hawking's simple discussion based on the area theorem of black hole event horizon. Here we recall the original discussion in Reference[1].

We simply consider two spin-less black holes with mass M_1 and M_2. In head-on collision of the spin-less black holes, the final state will be also a spin-less black hole with mass M_{tot}. The energy E converted into the gravitational radiation is related to the mass parameters by the energy conservation

$$M_1 + M_2 = M_{tot} + E. \tag{1}$$

On the other hand, we have the area theorem of black hole event horizon[8] which states that the area of the event horizon never decreases between two not intersecting spatial hypersurfaces under certain conditions. Then we have an inequality between initial and final area of black hole horizon,

$$M_1^2 + M_2^2 \leq M_{tot}^2, \tag{2}$$

if we approximate the black holes by Schwarzschild black hole. The equator would be attained if two stationary black holes instantaneously merged into a final stationary black hole. Consequently, the relations (1) and (2) restrict the energy converted into the gravitational radiation as

$$E < M_1 + M_2 - \sqrt{M_1^2 + M_2^2}. \tag{3}$$

Especially in the case of identical masses $M_1 = M_2$, the energy rate of the gravitational radiation $E/2M_1$ should be less than 29%.

After that work, numerical simulations[3] have shown in the identical mass head-on collision the energy rate of the gravitational radiation (∼0.1%) is two orders of magnitude smaller than Hawking's bound. Anyway, that means the evaluation of the area theorem by Schwarzschild black hole horizon is not incorrect. On the other hand, the upper bound is too large to predict anything for the collision. In a sense, it is not realistic to evaluate the event horizon area of colliding black hole by its gravitational mass as $16\pi M_i^2$, since the event horizon is determined by not only local geometry around the black hole but also global geometry including the other black hole. How can we turn the upper bound severe by improving the evaluation of the event horizon area? It is the main aim in the present article to clarify why gravitational radiation is not much in such head-on collision of black holes, by our improved evaluation of the area theorem in inequality (2).

The key idea of the present work is that the event horizon area of just merging black hole is far different from that of Schwarzschild horizon. Roughly speaking, the event horizon (EH) is extended from the Schwarzschild horizon to the direction of the collision, since a light ray emanating into this direction will be affected also by the gravitation of the opponent black hole. It becomes harder for the light ray to escape from the gravitation to the future null infinity. The event horizon of the colliding black holes is illustrated in Fig. 1.

The difference ΔA_i between EH area and $16\pi M_i^2$, which is the area of a corresponding Schwarzschild black hole with mass M_i, should be included into the area theorem eq.(2) as

$$M_1^2 + M_2^2 + \frac{\Delta A_1 + \Delta A_2}{16\pi} \leq M_{tot}^2. \tag{4}$$

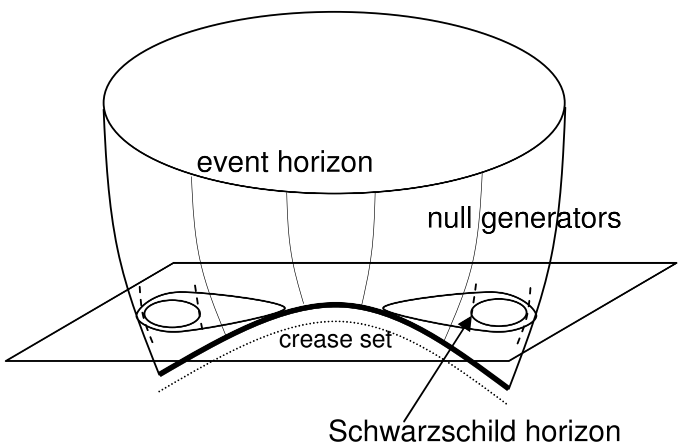


Figure 1: An event horizon of the coalescing black holes is illustrated. The event horizon is extended into the direction of the collision. Moreover a line-like crease set appears.

For simplicity, we consider head-on collision of spin-less black holes with identical masses $M_1 = M_2$, which finally settles to a Schwarzschild black hole with mass M_{tot}. Since two black holes are identical, the excesses of the EH area over the Schwarzschild horizon area are also identical $\Delta A_{tot} = \Delta A_1 + \Delta A_2 = 2\Delta A_1$. From the conservation low $M_1 + M_2 = M_{tot} + E$ and the area theorem eq.(4), we obtain another upper bound of the energy rate,

$$2M_1^2 + \frac{1}{8\pi}\Delta A_1 \leq (2M_1 - E)^2 \tag{5}$$

$$\Rightarrow \quad \frac{E}{2M_1} \leq 1 - \sqrt{\frac{1}{2} + \frac{\Delta A_1}{32\pi M_1^2}}. \tag{6}$$

In the next section, we estimate the area difference ΔA_1 from thermodynamical consideration.

3 thermodynamical estimation

In this section, we will estimate the area difference between EH in the black hole coalescence and Schwarzschild horizon, from black hole thermodynamical discussion. To consider thermodynamical aspects of the black hole coalescence, we adopt an analogy with a gas cylinder in uniform gravitation as a toy model of a merging black hole.

3.1 merging entropy

The work of Bekenstein[7] and Hawking[4] has shown, and that of many others confirmed, that the mechanical laws governing classical systems containing black holes can be placed in analogy with those of

thermodynamics. Further, the resulting correspondence between mechanical black hole variables (horizon area, surface gravity, etc.) and thermodynamic variables (entropy, temperature, etc.) has independent physical meaning when quantum mechanics is taken into account. This correspondence has been made explicit for a number of examples. Especially the important relation for the present work is that of the entropy and horizon area, $S = A/4$.

An upper bound of gravitational radiation for coalescence of black holes is derived by Hawking[1] based on the area theorem of the black hole horizon[8]. On the other hand, from the viewpoint of black hole thermodynamics the area theorem plays the role of the second law of thermodynamics, that is, the area theorem insists the entropy increases in the transition from an initial stationary black hole to a final stationary black hole.

On the stationary spin-less black hole with mass M, its EH coincides with Schwarzschild horizon whose area will be $16\pi M^2$ in Schwarzschild coordinate. Nevertheless when two black hole EHs are just merging, the black holes are no longer than stationary and their EHs will be different from the Schwarzschild horizon so that the EH area of a black hole with mass M_1 is estimated by $16\pi M_1^2 + \Delta A_1$ in which ΔA_1 will be brought by the gravitation of the other black hole with mass M_2.

From the viewpoint of black hole thermodynamics, the area differences ΔA_i correspond to entropy increase $\Delta S_i = \Delta A_i/4$ between initial stationary black holes and a final stationary black hole. In Hawking's discussion, the total area was estimated by the summation of each area of individual Schwarzschild black hole as $16\pi(M_1^2 + M_2^2)$. That may thermodynamically imply to omit the effect of mixing. In other words, this is to assume that two stationary merging black holes instantaneously settle into one final stationary black hole without changing its intrinsic configuration. The implication of the black hole uniqueness, however, that the final stationary black hole has forgotten its formation history means that the final black hole has suffered any mixture of the intrinsic configuration, and then there would be any entropy increase in this merging process. Therefore we expect that we are able to evaluate area difference ΔA_1 from the entropy increase ΔS_1.

Though the above discussion resembles to that of Gibbs' paradox for the mixing entropy, we are careful about difference between the ordinary mixing entropy and the present increase of black hole entropy. In Gibbs' paradox if two substances are identical, merged system immediately becomes equilibrium and there is no entropy increase of mixing. On the other hand for the black hole coalescence, even if the two black holes are identical, merged black hole does not settles to stationary (equilibrium) state until it forgets its own formation history, and then will suffer an entropy increase which we call merging entropy of the black holes.

The way to calculate such an entropy increase should be essentially based on state counting in statistical mechanics. That microscopic nature of such a dynamical black hole, however, has not been clarified yet. Rather an easy way to estimate the entropy increase upon the merging (mixing) of black holes may be to consider a toy model of a merging (mixing) thermodynamical black hole affected by the gravitational force of the other black hole.

The entropy of mixing for gas in chambers may be calculated by Gibbs' Theorem which states that when two different substances mix, the entropy increase upon mixing is equal to the entropy increase that would occur if the two substances were to expand alone into the mixing volume[2]. Then by analogy with the mixing gas, the merging entropy of black holes is expected to be also evaluated by similar individual expansion into a mixing volume. Since the entropy increase of merging black holes would be brought by a black hole getting near the opponent black hole, its substance will diffuse into the direction of the opponent black hole at that time.

Here we intuitively assume that the substances are bounded not by wall rather by gravitational force around the black hole. Then we estimate the merging entropy for black holes by analogy with ideal gas in the gravitational potential related to the black holes. Instead of expanding into the mixing volume, a substance will diffuse into intermediate space between the two black holes by balance of gravitation of two colliding black holes. Since that is one-dimensional force balance, in a toy model we simply imagine uniform gravitational force caused by each black hole. Initially there is only one black hole, and at coalescence, there arises the other black hole so that two uniform gravitational forces with equal strength and opposite directions, which are caused by the two black holes, offset each other and substances diffuse into the intermediate space between the two black holes.

[2]In this sense, the term "entropy of mixing" is a misnomer, since the entropy increase is not due to any "mixing" effect.

Suppose a gas cylinder with vertical length L (to be a distance between the two black holes) and with area of section s, is in a uniform gravitation. We will estimate the entropy increase by calculating the entropy difference between the ideal gases in the uniform gravitation and not in.

The chemical potential at height h of an ideal gas in a uniform gravitational acceleration g and temperature $\tau = k_B T$ is given by

$$\mu(h) = \tau \log \frac{n(h)}{n_q} + mgh, \tag{7}$$

where m is the mass of particle and $n(h)$ is its number density at height h, n_q is given by $(mV/2\pi\hbar^2)^{3/2}$ with dimensions of number density, and V is the volume of the chamber. In chemical equilibrium $\mu(h) = \mu(0)$, the number density and total number N is given by

$$n(h) = n(0) \exp\left(\frac{-mgh}{\tau}\right), \tag{8}$$

$$N = s \int_0^L n(h) dh = \frac{n(0)\tau s}{mg} \left[1 - e^{-mgL/\tau}\right]. \tag{9}$$

From Gibbs-Duhem relation and chemical equilibrium eq.(8), the entropy density $\sigma(h)$ is given by

$$\sigma(h) = -n(h)\left(\frac{d\mu}{d\tau}\right) + \left(\frac{dp}{d\tau}\right) \tag{10}$$

$$= -n(h)[\log(n(0)/n_q) - 5/2] \tag{11}$$

$$= \sigma(0)(n(h)/n(0)), \tag{12}$$

substituting the pressure p by $p = n\tau$. Then total entropy is given by

$$S = s \int_0^L \sigma(h) dh = \sigma(0) s \int_0^L n(h) dh = \sigma(0) N/n(0). \tag{13}$$

At the merging of two black holes the uniform gravitational acceleration g would be canceled out by the other uniform gravitational acceleration $-g$ of the opponent black hole. Then without the gravitational force, uniform number density $n' = N/sL$ and entropy of the uniform ideal gas are related by $S' = -N[\log(n'/n_q) - 5/2]$. Therefore the entropy increase by the offset of the uniform gravitation is given by

$$\Delta S = S' - S = N\left[-\log(n'/n_q) + \log(n(0)/n_q)\right]] \tag{14}$$

$$= N \log \frac{n(0)sL}{N} \tag{15}$$

$$= N \log \frac{mgL}{\tau} \frac{1}{1 - e^{-mgL/\tau}}. \tag{16}$$

$$\tag{17}$$

Assuming that m, $mgL << \tau$, which is justified with a large number of particles, eq.(17) accepts following approximation

$$\Delta S \sim N \log \frac{mgL}{\tau} \frac{1}{\frac{-mgL}{\tau}(-1 + \frac{mgL}{2\tau})} \tag{18}$$

$$\sim N \log(1 + \frac{mgL}{2\tau}) \tag{19}$$

$$\sim \frac{NmLg}{2\tau} = \frac{NmLg}{2k_B T}. \tag{20}$$

Now we evaluate this formula for one of colliding black holes. The temperature in the chamber would be considered to be Hawking temperature $1/8\pi M_{tot}$ of the final black hole. The height of the cylinder chamber L will roughly be a distance between the two black holes. In the present work, we evaluate

this distance by the length of the crease set l[10][9][12]. The crease set is the set of past endpoint of null horizon generator and an acausal subset as shown in Fig.1. As studied in [12][9][13], the length of the crease set is useful to evaluate the initial separation between colliding black holes, since that is a covariant value of the distance determined by causal structure. Especially the topology of event horizon strongly depends on the crease set and its timeslicing[9]. Indeed, the black hole head-on collision is due to the line-like crease set as illustrated in the figure 1. When we take a natural timeslicing for the black hole coalescence as in the figure 1, the spacetime points where two black holes are formed, are connected by the crease set. Therefore the length of the crease set l would represent the distance between the two black holes as $L = l$.

Moreover, g is roughly estimated by the surface gravity of the black hole $g = 1/4M_1$. Though Nm would be the mass of matter constituting the black hole, it does not seem that all of the matter takes part in this process. Then we speculate that only substances in the direction to the opponent black hole will take part in this merging process, as illustrated in Fig.2. Considering six directions which are along positive and negative x-, y-, z-axis, we schematically suppose that such substances in the direction to the opponent black hole are along the positive x-axis. Since that is a sixth of the whole directions around the black hole, the total mass Nm contributing to this process would be given by $M_1/6$.

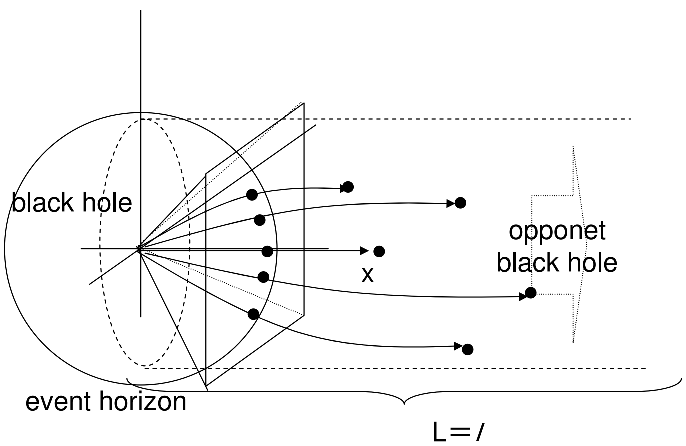


Figure 2: Only 'particles' in a positive x-directions diffuse into the directions of the opponent black hole. They would be a sixth of the substances of the black hole.

Hence the black hole merging entropy is evaluated as

$$\Delta S_1 = \frac{1}{2}\frac{M_1}{6}l\frac{1}{4M_1}(8\pi M_{tot}) = \frac{\pi}{6}M_{tot}l. \tag{21}$$

Total entropy increase is given by summing up the contributions of two black holes as $2\Delta S_1$.

3.2 Energy rate of gravitational radiation with merging entropy

Now we estimate the energy rate of gravitational radiation by the area theorem of the event horizon incorporating the merging entropy. Substituting the merging entropy (21) into the area theorem (6) with the Bekenstein-Hawking relation $\Delta A_1 = 4\Delta S_1$, we have an inequality

$$\frac{E}{2M_1} < 1 - \sqrt{\frac{1}{2} + \frac{M_{tot}l}{48M_1^2}}. \tag{22}$$

Here we note that if we believe the hoop conjecture[11] the length of the crease set might be bounded from above. The hoop conjecture forbids such anisotropic event horizon that any hoop with length $4\pi M$ cannot surround it. On a certain spatial hypersurface (see Fig.3)[12], a long crease set implies the existence of long spindle-like event horizon forbidden in the hoop conjecture.

In the case of low-energy collision, from the hoop conjecture we evaluate the length of the crease set l by the maximum value $l_{max} = 2\pi M_{tot}$ in order for a hoop with length $4\pi M_{tot}$ to circulate the spindle-like total black hole as shown in Fig.3. For the total mass $M_{tot} = 2M_1 - E$, and the energy rate $x = E/2M_1 < 1$, the inequality (22) becomes

$$x < 1 - \sqrt{\frac{1}{2} + \frac{\pi x}{6}(1-x)^2} \tag{23}$$

$$\Rightarrow \quad x < 1 - \sqrt{\frac{1}{2}\frac{1}{1-\frac{\pi}{6}}} \sim -0.024. \tag{24}$$

Consequently we see gravitational wave is hardly radiated in this most severe situation.

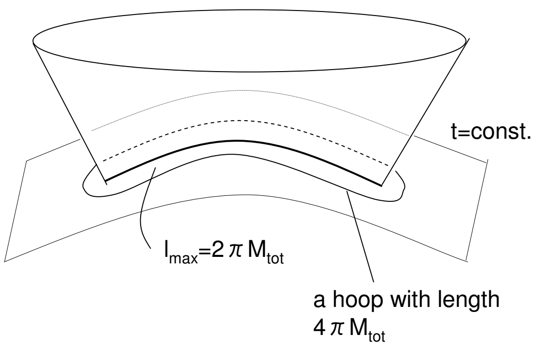


Figure 3: The crease set implies a spindle-like black hole on an appropriate spatial hypersurface. The crease set should be shorter than $2\pi M_{tot}$ so that a hoop with a length $4\pi M_{tot}$ can circulate the spindle-like black hole.

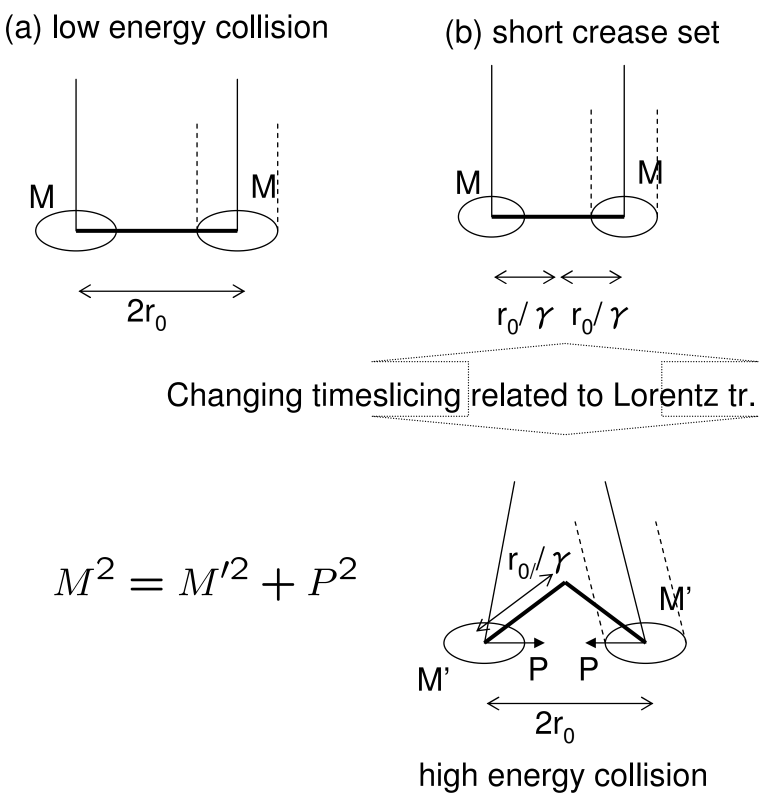
Next we consider the case of high-energy collision of black holes. To estimate the energy rate of the high-energy collision, we will discuss the length of crease set again. Here we comment that the distinction between low- and high-energy collisions depends on timeslicing. By a coordinate transformation related to the Lorentz boost of the black holes, the irreducible mass M_{irr} of the black hole can be converted into the kinetic energy brought by linear momentum P while the ADM-energy of the colliding system does not change. In the following we show that low-energy collision with identical masses M_{irr} corresponds to high-energy collision with identical masses $M'_{irr} = \sqrt{M_{irr}^2 - P^2}$ and linear momentum P, by an appropriate coordinate transformation which changes the coordinate separation between the colliding black holes.

Firstly, to develop the same discussion between low- and high-energy collision, we prepare two pair of coalescing black holes whose crease sets are with length $2r_0$ ~Fig.4(a) and with $2r_0/\gamma$ ~Fig.4(b), where their crease sets are almost tangent to their timeslicing, respectively. To make coordinate separations between each colliding black holes same, we changes the timeslicing of collision (b) related to the Lorentz boost with factor $\gamma > 1$ as illustrated in Fig.4(b).

Then each black hole in (b) gains linear momentum (Bowen-York parameter[15]) $P = \sqrt{\gamma^2 - 1}M_{irr}$ and their irreducible mass M_{irr} decreases by factor $1/\gamma$ since 'ADM-energy' $M_{ADM}^2 = M_{irr}^2 + P^2$ does not change. After all we see that when we compare two colliding systems with same ADM-energy and same coordinate separation the crease set of high-energy collision is shorter than that of low-energy collision in proper length.

Here it should be noted, however, even in very high-energy collision, the crease set cannot get as much shorter as possible. Black holes too close to each other will not be regarded as the colliding of two black holes, because before their coalescence two black holes may already have been inside of a single black hole horizon. Then it seems that we cannot have such an initial conditions in vain. We will estimate the minimal length of crease set for the very high-energy collision also by the hoop conjecture from this viewpoint.

If two black holes are close to each other and their common crease set is shorter than $l_m = \pi(M_1 + M_2) = 2\pi M_1$, there would be a hoop surrounding the two black holes in a shape of connected two half circle (see Fig.5) and with length $\pi(2M_1) + \pi(2M_1) + 2l_m = 8\pi M_1 = 4\pi(M_1 + M_2)$, which is the length



$$M^2 = M'^2 + P^2$$

Figure 4: On (a), the event horizon in the low-energy head-on collision is illustrated. The proper length of the crease set is r_0 and its total energy is M. On (b), we prepare an event horizon which is similar to the event horizon in (a) but its crease set is with length r_0/γ. By coordinate transformation related to the Lorentz boost with factor γ, the coordinate separation becomes same as that of (a). That corresponds to the event horizon in the high-energy head-on collision as illustrated.

of initial total mass $(M_1 + M_2)$ multiplied by 4π. On a spatial hypersurface containing the crease set, the hoop conjecture implies that a common black hole horizon containing both two colliding black holes will have already existed. Since this is not appropriate for initial condition for black holes to collide, we conclude $l > l_{min} = 2\pi M_1$ for the high-energy black hole coalescence.

Hence, we evaluate the length of the crease set by $l = 2\pi M_1$ for maximally boosted black hole collision. The energy rate $x = E/2M_1$ is given by the area theorem (22) as

$$\frac{E}{2M_1} < 1 - \sqrt{\frac{1}{2} + \frac{\pi M_{tot}}{24 M_1}} \tag{25}$$

$$x < 1 - \sqrt{\frac{1}{2} + \frac{\pi}{12}(1-x)} \tag{26}$$

$$\Rightarrow \quad x < 1 - \frac{\pi}{24} - \sqrt{\frac{1}{2} + \frac{\pi^2}{576}} \sim 0.15. \tag{27}$$

This upper bound is fairly close to an already given value 0.14 of high-energy limit in the numerical work[5] extrapolated by ZFL-PP calculation[16][17] and the prediction of linear perturbation 0.16 by D'Eath and Payne in Ref.[6].

The area difference ΔA from the merging entropy ΔS explains not only the numerical result of low-energy collision but also the independent result of numerical simulation for high-energy collision. That implies the dynamical effect of collision is always represented by this entropy increase of merging effect.

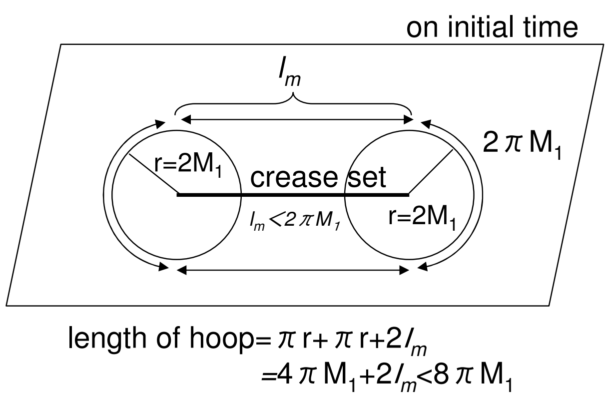


Figure 5: Two half circles are connected as shown. Since this connected half circles with length $8\pi M_1$ circulates both colliding Schwarzschild horizons, initially there are not two colliding black holes but is only one merged black hole.

Furthermore we investigate γ-factor dependence in small β limit by the Lorentz contraction of the crease set. From inequality (22) and $l = l_{max}/\gamma = 2\pi M_{tot}/\gamma$, the energy rate is bounded by

$$\frac{E}{2M_1} < 1 - \sqrt{\frac{1}{2} + \frac{\pi M_{tot}^2}{24 M_1^2}\frac{1}{\gamma}} \tag{28}$$

$$\Rightarrow \quad x < 1 - \sqrt{\frac{1}{2}\frac{\gamma}{1-\frac{\pi}{6}}} \sim -0.02 + 0.3\beta^2 + 0.1\beta^4 .... \tag{29}$$

Though there is no correspondence to the formula of ZFL-PP approximation in Ref.[16][17], that may agree with the numerical results in the range around small β within a statistical accuracy. On the other hand, γ-dependence in the large γ limit is hard to estimate since the length of the crease set should be saturated at $l = 2\pi M_1$ by above intuitive discussion.

4 conclusion and discussions

We have studied the amount of energy converted into gravitational radiation in black hole head-on collision, revising Hawking's discussion by incorporating the entropy increase by merging. In the head-on collision of identical mass black holes, introducing the merging entropy which resembles mixture entropy, area theorem of the event horizon is re-evaluated. Incorporating the re-evaluated area theorem to Hawking's original discussion to have an upper bound of energy rate of the gravitational radiation, the upper bound is lowered from Hawking's 29% to negligible ($\sim 0\%$) in low-energy collision and to 15% in maximally high-energy limit. These results well agree with two independent numerical simulations which are 0.1% in low-energy[3] and 14% in maximally high-energy limit[5], at once. Of course, as we have avoided precise estimation, the upper bounds would be typical values rather than strict upper bounds.

In the present analysis, the contribution of the merging entropy implies the merging process is an invertible process. That is consistent with the fact that black holes can merge but never split. In the black hole dynamics, is there another invertible process? If so, we will find another component of the black hole entropy like the merging entropy. By contraries, one may speculate that much gravitational radiation emitted during the transition from toroidal black hole to a spherical black hole, since such process is topologically revertible[9][18]. For, it is not expected that any dynamical component of the black hole entropy lower the upper bound of the energy rate in such a revertible process. Moreover that might be match with the fact that in circular orbiting coalescence of the black holes more gravitational radiation can be emitted, because the circular orbit of coalescing black holes would be regarded as the formation and collapse of the toroidal black hole[19].

In microscopic scales described by quantum gravity, it might be possible for a black hole to split. Are there any failures of our analysis in that sub-Planckian geometry? This is probably understood as to be the following. The area theorem of black hole requires an energy condition. Since in such microscopic situations quantum effect will violate the energy condition, our analysis of the merging entropy loses its foundation and is consistent with the fact that this process becomes revertible.

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